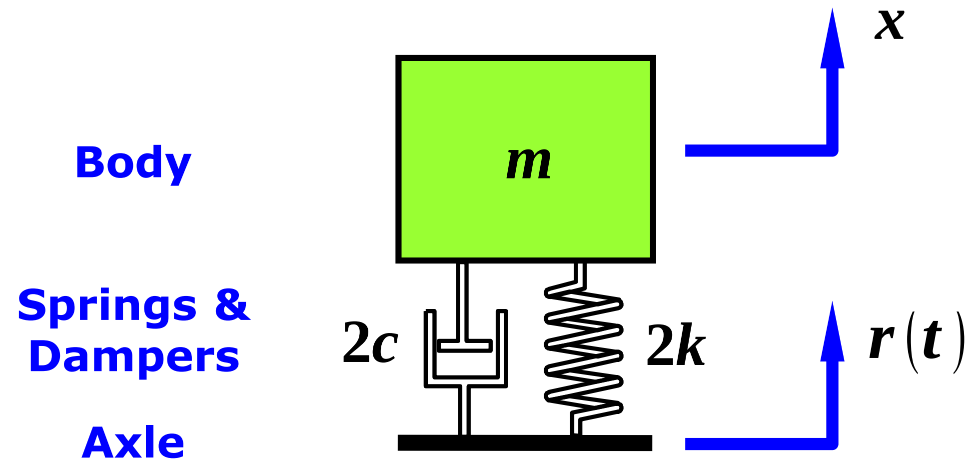
Dynamics
MM2DYN

Worked Examples on Harmonic Excitation

1. Single-axle caravan



Equation of motion:

$$M\ddot{z} + C\dot{z} + Kz = C_r\dot{r}(t) + K_r r(t)$$

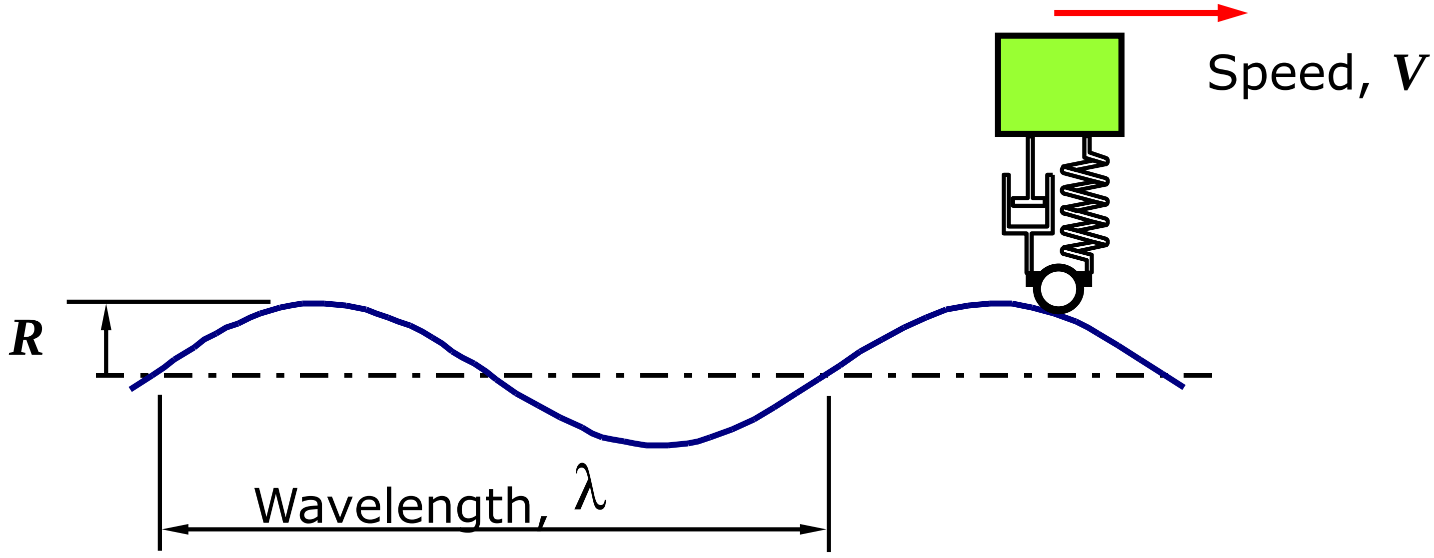
$$\boxed{m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2kr(t)}$$

Suppose the road profile is sinusoidal, so that the displacement input is $r(t) = R\sin\omega t$

Q1. How does suspension stiffness affect the response of the caravan?

Q2. Does vehicle speed affect the response?

Q3. How important are the dampers?



Time to travel one wavelength, $T = \dfrac{\lambda}{V}$ s

Excitation frequency, $\omega = \dfrac{1}{T} = \dfrac{V}{\lambda}$ Hz

$= \dfrac{2\pi V}{\lambda}$ rad/s

Equation of motion: $\boxed{m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2k\,r(t)}$ (1)

Substitutions: $r(t) = R\,e^{\mathbf{i}\,\omega\,t}$, $\dot{r}(t) = i\omega R\,e^{\mathbf{i}\,\omega\,t}$ and $x(t)_{SS} = X^*\,e^{\mathbf{i}\,\omega\,t}$, $\dot{x}(t)_{SS} = i\omega X^*\,e^{\mathbf{i}\,\omega\,t}$, $x(t)_{SS} = -\omega^2 X^*\,e^{\mathbf{i}\,\omega\,t}$

Hence, $$\left(\left(2k - m\omega^2\right) + \mathbf{i}\,2c\,\omega\right)X^*\,\cancel{e^{\mathbf{i}\,\omega\,t}} = \left(2k + \mathbf{i}\,2c\,\omega\right)R\,\cancel{e^{\mathbf{i}\,\omega\,t}}$$

and $$\boxed{X^* = \frac{\left(K_r + \mathbf{i}\,C_r\,\omega\right)R}{\left(K - M\omega^2\right) + \mathbf{i}\,C\omega} = \frac{\left(2k + \mathbf{i}\,2c\,\omega\right)R}{\left(2k - m\omega^2\right) + \mathbf{i}\,2c\,\omega}}$$

$\boldsymbol{X^*}$ is in the form $$\frac{c + \mathbf{i}\,d}{e + \mathbf{i}\,f}$$

The amplitude of the response is

$$\left|X^*\right| = \frac{\sqrt{c^2 + d^2}}{\sqrt{e^2 + f^2}} = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\;R}{\sqrt{\left(K - M\omega^2\right)^2 + C^2\omega^2}} = \frac{\sqrt{4k^2 + 4c^2\omega^2}\;R}{\sqrt{\left(2k - m\omega^2\right)^2 + 4c^2\omega^2}}$$

You can also show these in terms of γ, ω, and ω_n

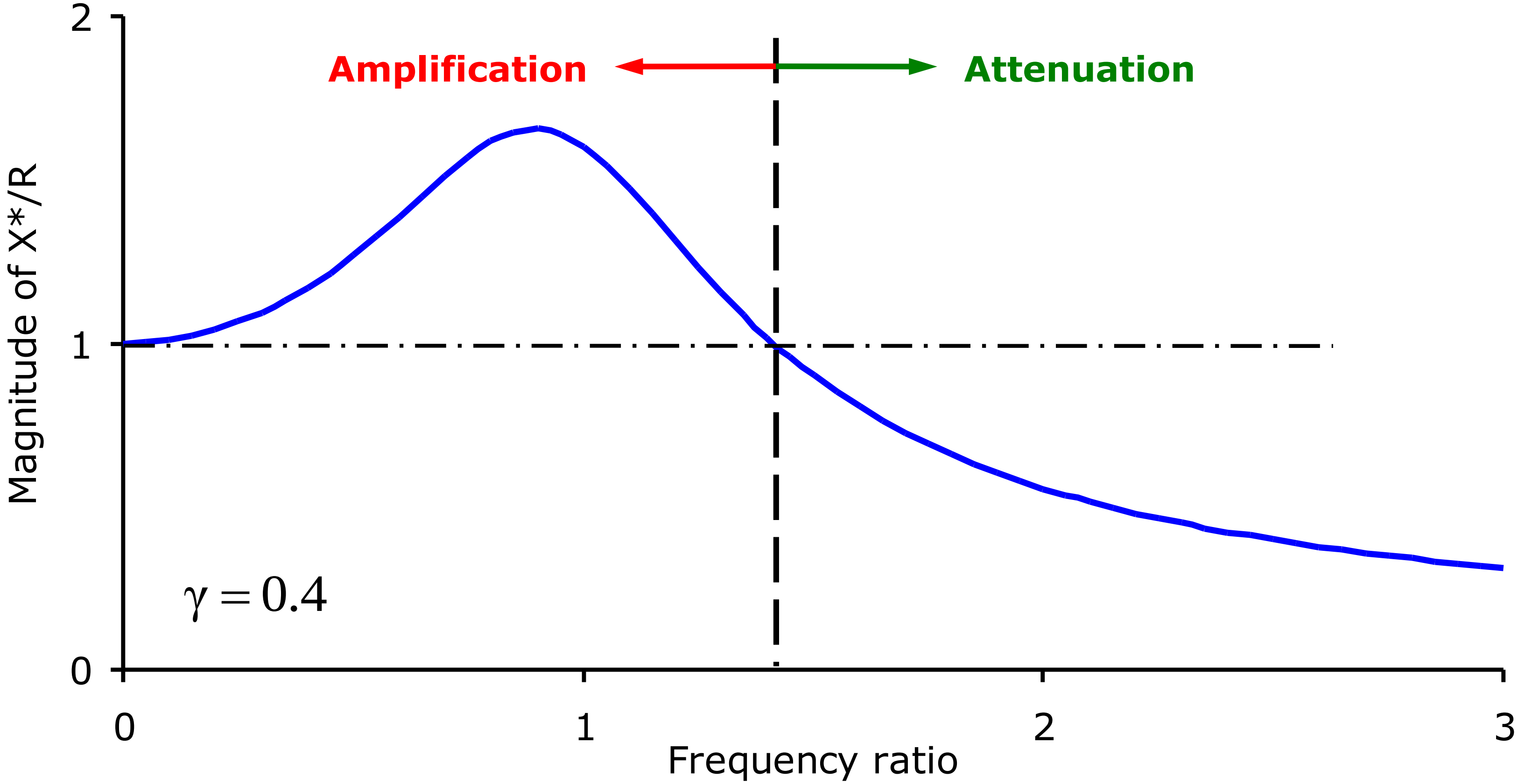
$$x(t)_{SS} = \left|X^*\right| \cos(\omega t + \alpha)$$

$$\left|X^*\right| = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\;R}{K\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\frac{\omega^2}{\omega_n^2}}} = \frac{\sqrt{1+4\gamma^2\frac{\omega^2}{\omega_n^2}}\;R}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\frac{\omega^2}{\omega_n^2}}}$$

$$\alpha = \tan^{-1}\left(\frac{C_r\omega\left(K - M\omega^2\right) - K_r C\omega}{K_r\left(K - M\omega^2\right) + C_r C\omega^2}\right)$$

$$\alpha = \tan^{-1}\left(\frac{-cm\omega^3}{k\left(2k - m\omega^2\right) + 2c^2\omega^2}\right) = \tan^{-1}\left(\frac{-2\gamma\left(\frac{\omega}{\omega_n}\right)^3}{\left(1-\frac{\omega^2}{\omega_n^2}\right) + 4\gamma^2\frac{\omega^2}{\omega_n^2}}\right)$$

The caravan's displacement depends on the excitation frequency, which is proportional to the vehicle speed



The vehicle suspension is an example of **vibration isolation**, which we will look at in more detail later in the module

Returning to the original questions:

- **Q1.** How does suspension stiffness affect the response of the caravan?
- **Q2.** Does vehicle speed affect the response?
- **Q3.** How important are the dampers?

(A) What happens of the springs are very stiff?

$$\left|X^*\right| = \frac{\sqrt{4k^2 + \underbrace{4c^2\omega^2}_{\approx 0}}\; R}{\sqrt{\left(2k - \underbrace{m\omega^2}_{\approx 0}\right)^2 + \underbrace{4c^2\omega^2}_{\approx 0}}} = \frac{\sqrt{4k^2}\; R}{\sqrt{\left(2k\right)^2}}$$

This says that the **caravan follows the road profile exactly**

The result is **independent of road speed**

Motion would be very uncomfortable at high speeds

$$\left| X^{*} \right| = R$$

This says that **the caravan follows the road profile exactly**

The result is **independent of road speed**

Motion would be very uncomfortable at high speeds

What would **really** happen at high speeds?

The assumption that the tyres stay in contact with the road would cease to be valid

The caravan would bounce from one bump to the next

(B) What happens if the springs are very soft?

This would give a low natural frequency and provide good attenuation of road input at medium and high speeds to give a smooth ride

BUT

It would require a large spring displacement to support the weight of the vehicle

Can produce excessive roll on cornering

So, what stiffness should the designer choose?

High stiffness tends to improves road holding, but at the expense of ride comfort

Low stiffness tends to improves ride comfort, but at the expense of road holding

For any vehicle, the final selection will be a balance between the two

Q3. How important are the dampers?

What happens if you remove the dampers?

Does vehicle speed affect the response?

$$\left|X^*\right| = \frac{\sqrt{4k^2 + 4c^2\omega^2}\; R}{\sqrt{\left(2k - m\omega^2\right)^2 + 4c^2\omega^2}}$$

$c = 0$

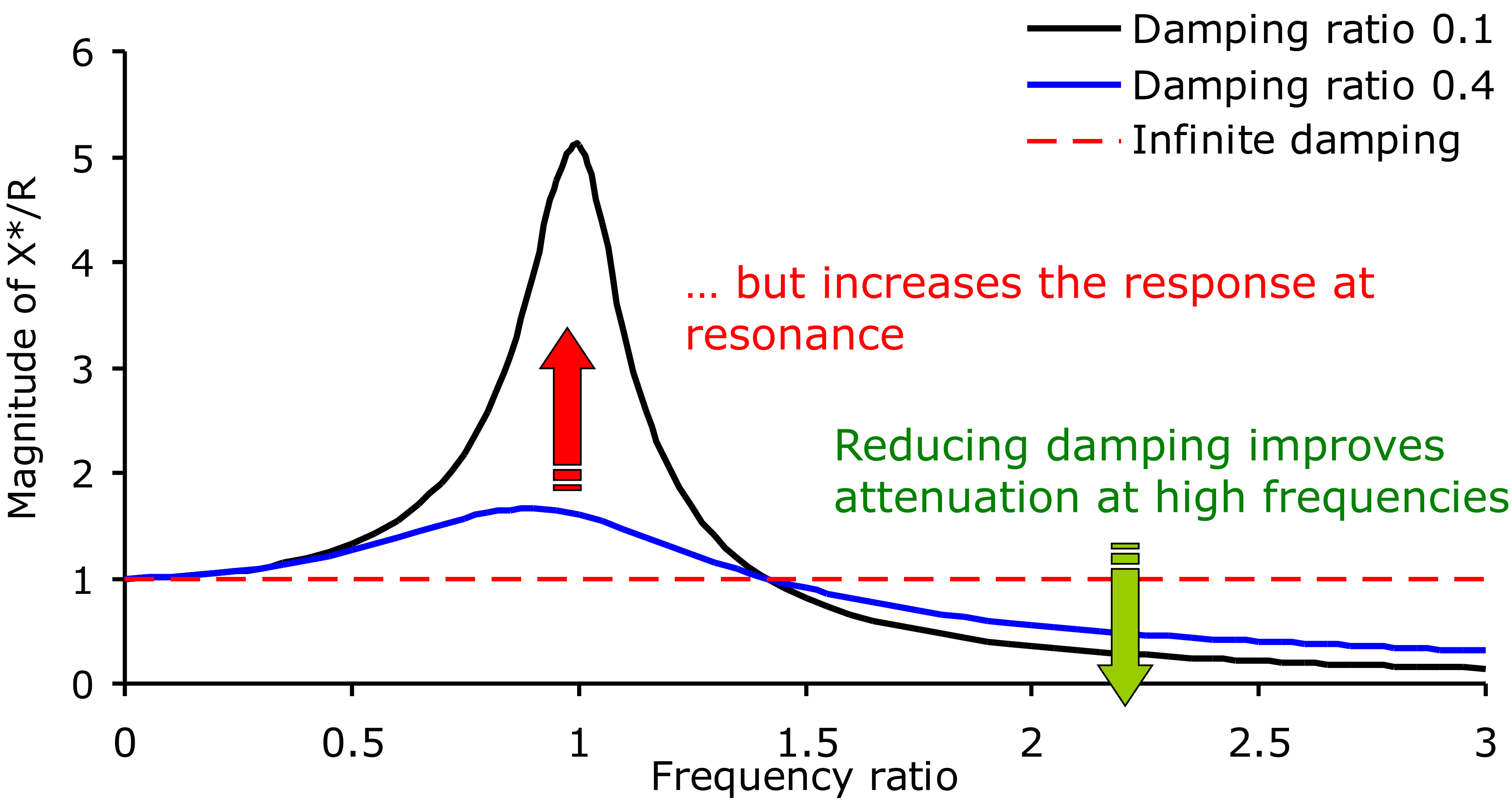
$c = 0$

If the vehicle speed is such that $2k - m\omega^2 = 0$, the amplitude of the caravan will be **infinite !**

What would **really** happen?

The suspension would become non-linear and prevent very large displacements

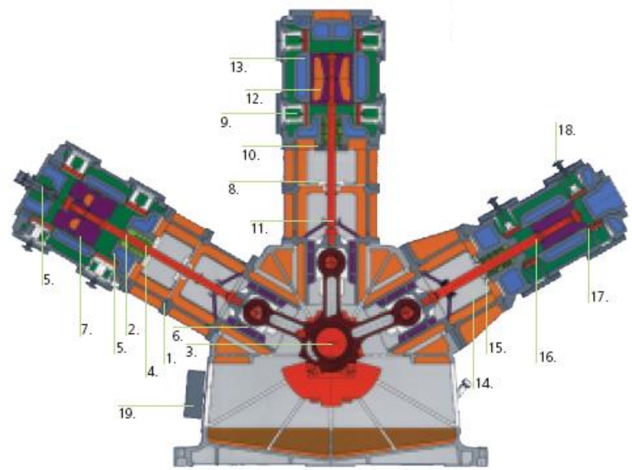
So, what damping level should the designer choose?



Another balance — but biased towards limiting the resonant response

2. Application of the Frequency Response Function

A reciprocating air compressor supported on a set of six resilient mounts



DATA

Mass, $\boldsymbol{m}$ = 4 tonne

Crankshaft speed, $\boldsymbol{\Omega}$ = 31.4 rad/s [300 rev/min]

Overall vertical stiffness of mounts, $\boldsymbol{k}$ = 2.5 MN/m

Damping ratio, $\boldsymbol{\gamma}$ = 0.04

The reciprocating acceleration of the pistons results in a vertical reaction force on the compressor body given by

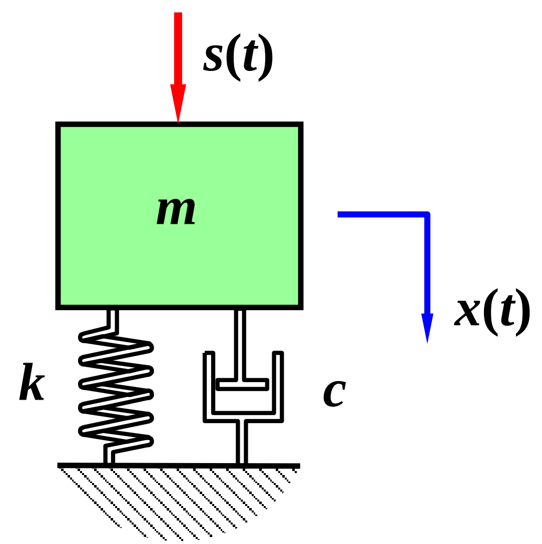
$$s(t) = S_1 \cos \omega_1 t + S_2 \sin \omega_2 t = 1.3\Omega^2 \cos \Omega t + 3.0\Omega^2 \cos 2\Omega t$$

There are **two frequency components**, one at the crank rotational frequency, $\omega_1 = \Omega$, and one at twice the crank frequency, $\omega_2 = 2\Omega$

Objective: **To find $x(t)$, the waveform of the steady-state vertical displacement of the compressor**

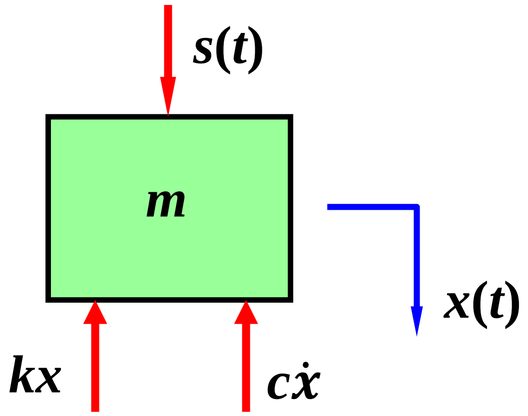
Solution method

STEP 1: Dynamic model



Solution method

STEP 2: FBD



Solution method

STEP 3: EOM

$$M\ddot{x} + C\dot{x} + Kx = s(t)$$

$$\boxed{M\ddot{x} + C\dot{x} + Kx = S_1 \cos\omega_1 t + S_2 \cos\omega_2 t}$$

Solution method

STEP 4: Consider the effect of the forcing function $s(t)$

- Each sinusoidal term in the excitation will produce a steady-state response that is sinusoidal with the same frequency as the excitation

- Once the response to each excitation term has been found, the total response is obtained by adding the two together

$$s(t) = S_1 \cos\omega_1 t \ + \ S_2 \cos\omega_2 t$$

$$x(t)_{SS} \ = \ X_1 \cos(\omega_1 t + \alpha_1) + X_2 \cos(\omega_2 t + \alpha_2)$$

To solve the problem, we just need the values for $\boldsymbol{X_1}$, $\boldsymbol{\alpha_1}$, $\boldsymbol{X_2}$ & $\boldsymbol{\alpha_2}$

We use the frequency response function (FRF) to do this

$$X_1^* = H(\omega_1) \times S_1 \qquad\qquad X_2^* = H(\omega_2) \times S_2$$

$\boldsymbol{X_1}$ $\boldsymbol{\alpha_1}$ $\boldsymbol{X_2}$ $\boldsymbol{\alpha_2}$

We derived an expression for the FRF for this system previously
(see page 4 of the handout)

$$H(\omega) = \frac{X^*}{S} = \frac{1}{m}\,\frac{1}{\left(\omega_n^2 - \omega^2\right) + \mathbf{i}\,2\gamma\,\omega_n\,\omega}$$

For each excitation term, we have $X_j^* = H\left(\omega_j\right) \times S_j$

so that $X_j = \left| X_j^* \right| = \left| H\left(\omega_j\right) \right| \times S_j$ where j = 1, 2

$$\left| X_j^* \right| = \frac{1}{m}\sqrt{\frac{1}{\left(\omega_n^2 - \omega_j^2\right)^2 + 4\gamma^2\,\omega_n^2\,\omega_j^2}} \times S_j$$

and the phase angles are

$$\alpha_j = \tan^{-1}\left(\frac{-2\gamma\,\omega_n\,\omega_j}{\omega_n^2 - \omega_j^2}\right)$$

Using the data given,

$\boldsymbol{\omega_1}$ = 31.4 rad/s $\quad$ $\boldsymbol{S_1}$ = 1,283 N $\quad$ $|\boldsymbol{H}(\boldsymbol{\omega_1})|$ = 6.81 x 10^{-4} mm/N

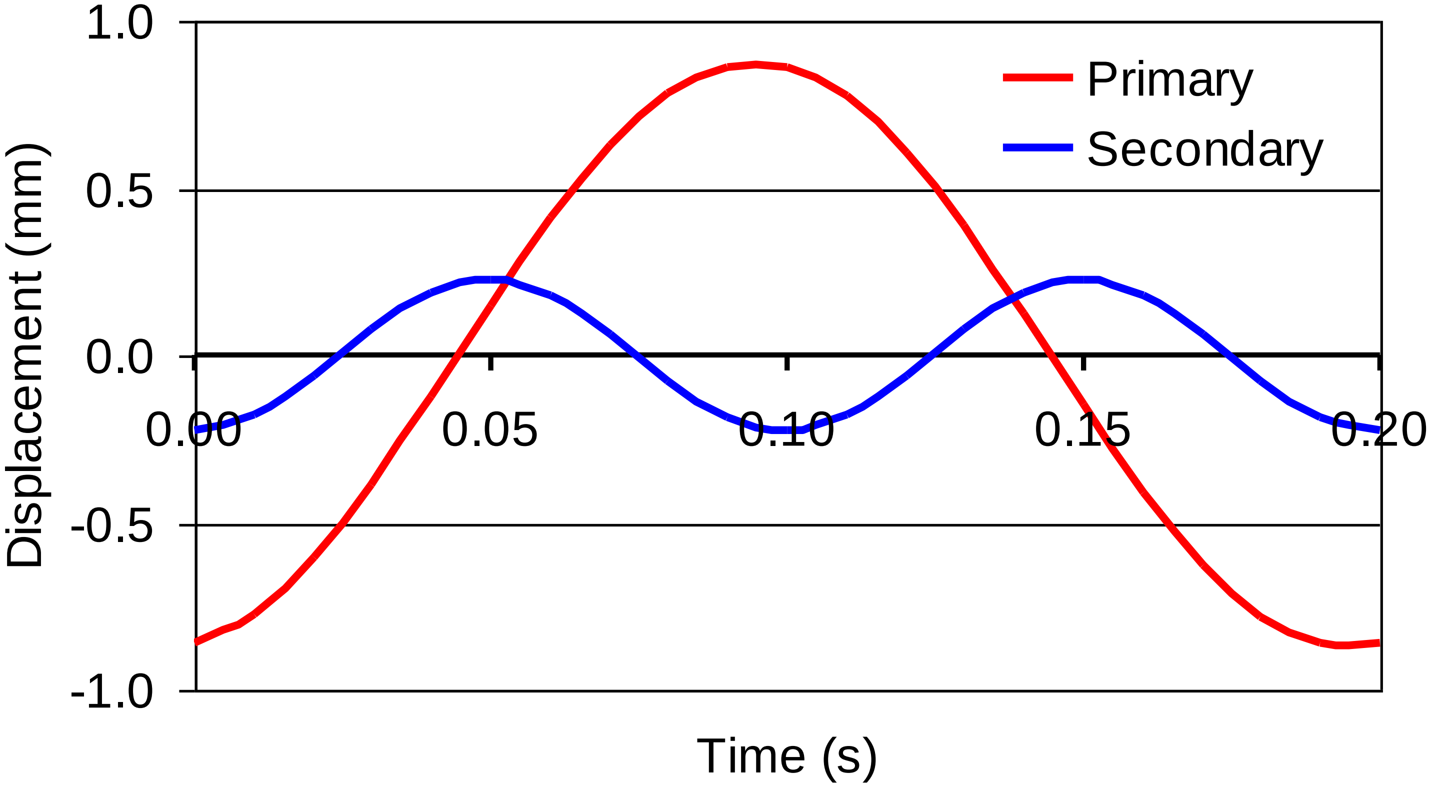
$\boldsymbol{X_1}$ **= 0.87 mm** $\quad$ $\boldsymbol{\alpha_1}$ **= -170°**

$\boldsymbol{\omega_2}$ = 62.8 rad/s $\quad$ $\boldsymbol{S_2}$ = 2,961 N $\quad$ $|\boldsymbol{H}(\boldsymbol{\omega_2})|$ = 7.52 x 10^{-5} mm/N

$\boldsymbol{X_2}$ **= 0.22 mm** $\quad$ $\boldsymbol{\alpha_1}$ **= -178°**

and remembering…

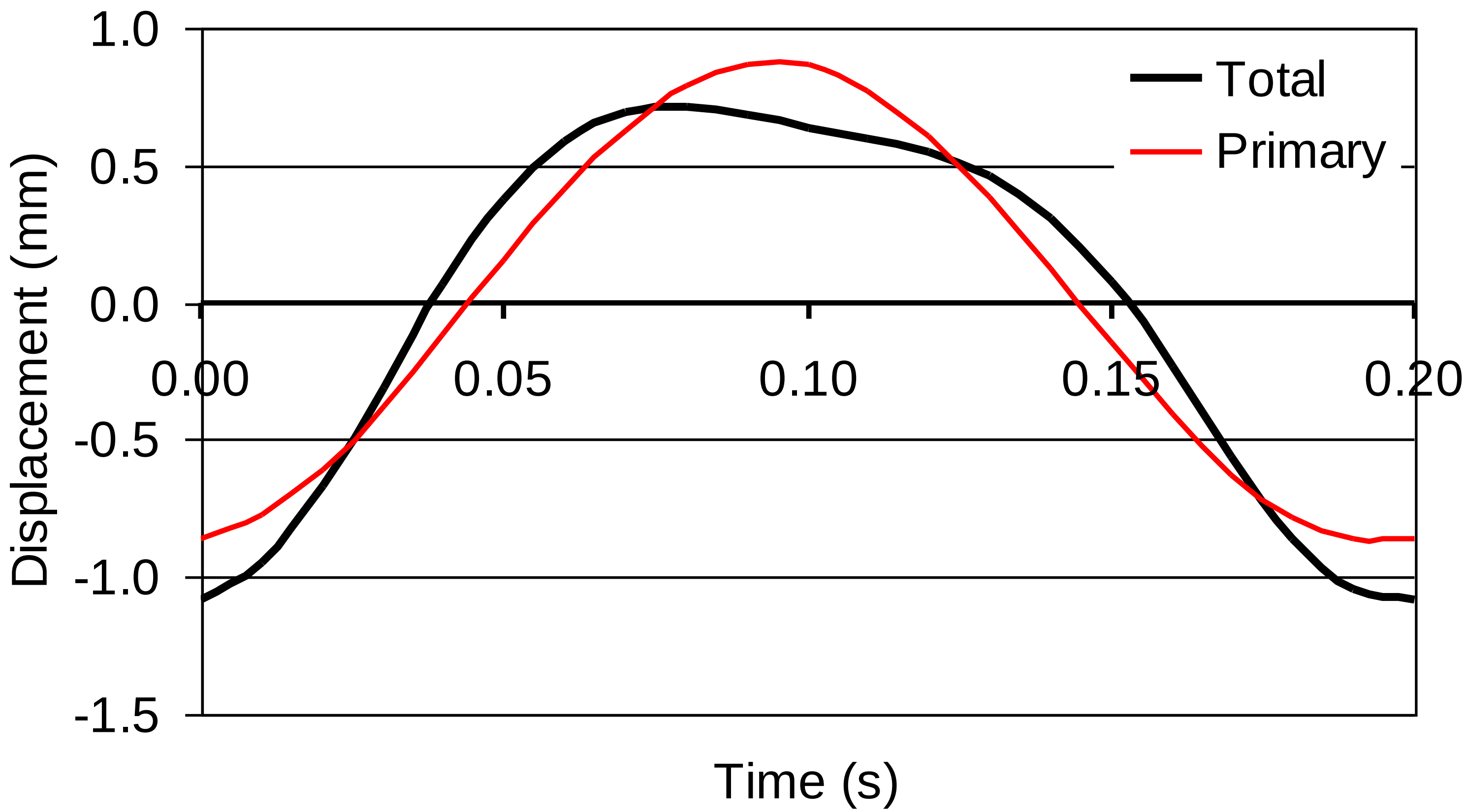
$$x(t)_{SS} = X_1 \cos\left(\omega_1 t + \alpha_1\right) + X_2 \cos\left(\omega_2 t + \alpha_2\right)$$



"Primary": ω_1 **= 5 Hz** X_1 **= 0.87 mm** α_1 **= -170°**

"Secondary": ω_2 **= 10 Hz** X_2 **= 0.22 mm** α_1 **= -178°**

The total response is obtained by adding the individual responses together


1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.05
0.10
0.15
0.20
Displacement (mm)
Time (s)
Total
Primary

NEXT